VIII. Applications of Convex Surfaces

A. Bennequin type inequalities

we first prove an inequality for closed surfaces

Th$^{\underline{m}}$ 1:

> (M,ξ) a tight contact manifold
> Σ a surface in M
> if $e(\xi)$ is the Euler class of ξ, then
> $$|\langle e(\xi),[\Sigma]\rangle| \begin{cases} \leq -\chi(\Sigma) & \text{if } \Sigma \neq S^2 \\ = 0 & \text{if } \Sigma = S^2 \end{cases}$$

recall if v is a vector field in ξ that is transverse to the zero section then v^{-1}(zero section) is a 1-manifold $\gamma \subset M$

$e(\xi)$ is Poincaré dual to the homology class $[\gamma]$

so $e(\xi) \in H^2(M)$ and thus can be evaluated on elements of $H_2(M)$ like $[\Sigma]$

the value $\langle e(\xi),[\Sigma]\rangle$ is just the signed count of zeros of v on Σ (i.e. $\xi|_\Sigma$ is 4-dimensional the zero section and image of v are both 2-dimensional so take signed count of intersections)

exercise: the above inequality implies only a finite number of cohomology classes can be realized as the

Euler class of a tight contact structure

Hint: homology classes can be represented by closed surfaces

Proof: given $\Sigma \neq S^2$ make it convex

1) Γ is a union of S^1 (and $\chi(S^1)=0$)

2) no component of Γ bounds a disk by the Giroux criterion (Thm VII.10)

1) $\Rightarrow$ $\chi(\Sigma) = \chi(\Sigma_+) + \chi(\Sigma_-)$

exercise: $\langle e(\xi), [\Sigma] \rangle = \chi(\Sigma_+) - \chi(\Sigma_-)$

Hint: Consider a vector field directing Σ_ξ
It is a section of $T\Sigma$ and of ξ

so $\langle e(\xi), [\Sigma] \rangle - \chi(\Sigma) = -2\chi(\Sigma_-)$

≥ 0 since $\chi(\text{each component of } \Sigma_-) \leq 0$

$\therefore -\langle e(\xi), [\Sigma] \rangle \leq -\chi(\Sigma)$

similarly $-\langle e(\xi), [\Sigma] \rangle - \chi(\Sigma) = -2\chi(\Sigma_+) \geq 0$

so $\langle e(\xi), [\Sigma] \rangle \leq -\chi(\Sigma)$

$\therefore |\langle e(\xi), [\Sigma] \rangle| \leq -\chi(\Sigma)$ ✓

if $\Sigma = S^2$ the Giroux criterion says Γ connected

and so $S^2 \setminus \Gamma = D^2_+ \cup D^2_-$

$\therefore \langle e(\xi), [\Sigma] \rangle = 0$ ▦

Th$^{\underline{m}}$ 2 [Bennequin inequality]:

(M,ξ) a tight contact manifold

(1) if T a null-homologous transverse knot

then
$$sl(T) \leq -\chi(\Sigma)$$
for any Σ with $\partial\Sigma = T$

(2) if L a null-homologous Legendrian knot

then
$$tb(L) + |r(L)| \leq -\chi(\Sigma)$$
for any Σ with $\partial\Sigma = L$

Proof:

let L be a Lagrangian knot and

Σ be a Seifert surface for L

we first prove $tb(L) + r(L) \leq -\chi(\Sigma)$

to this end we positively stabilize L untill $tb(L) < 0$

note this does not change $tb(L) + r(L)$ so if

we can prove inequality for stabilized L (still denoted

by L) then we are done

since $tw_\xi(L, \Sigma) = tb(L) < 0$ we can make Σ convex

Th$^{\underline{m}}$ VII.5

Claim: $tb(L) + \chi(\Sigma_+) + \chi(\Sigma_-) = \chi(\Sigma)$

proof: Γ_Σ has $-tb(L)$ arcs (since $tb(L) = -\frac{1}{2}(\Gamma_\Sigma \cap L)$

and each arc has

2 end points by

Th$^{\underline{m}}$ VII.9)

call the arcs Γ_a

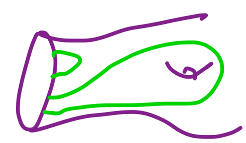
let $\Gamma_c = \Gamma_\Sigma - \Gamma_a$

$$\chi(\Sigma_+)+\chi(\Sigma_-) = \chi(\Sigma \setminus \Gamma_\Sigma)$$
$$= \chi((\Sigma \setminus \Gamma_a)\setminus \Gamma_c) \quad \text{since } \chi(\Gamma_c)=0$$
$$= \chi(\Sigma \setminus \Gamma_a) \quad \text{each arc has } \chi = 1$$
$$= \chi(\Sigma) - tb(L)$$

Claim: $tb(L) \leq -\chi(\Sigma_+)$

proof: since ξ tight no component of Γ_Σ bounds a disk

so there are at most $-tb(L)$ disk components in $\Sigma \setminus \Gamma_\Sigma$

eg  coming from arcs in Γ_Σ

$\therefore$ at most $-tb(L)$ disks in Σ_+

so $$\chi(\Sigma_+) = \#\text{disks} - \underbrace{\chi(\text{other components})}_{\leq 0}$$
$$\leq \#\text{disks} \leq -tb(L)$$

now
$$tb(L)+r(L) \leq tb(L)+r(L)-2tb(L)-2\chi(\Sigma_+)$$
$$= -tb(L)+r(L)-2\chi(\Sigma_+)$$
$$= -tb(L)+\chi(\Sigma_+)-\chi(\Sigma_-)-2\chi(\Sigma_+) \quad \text{Th}^{\text{m}}\ \text{VII}.9$$
$$= -tb(L)-\chi(\Sigma_-)-\chi(\Sigma_+)$$
$$= -\chi(\Sigma)$$

exercise: give similar proof that $tb(L)-r(L) \leq -\chi(\Sigma)$

now if T a transverse knot and Σ a surface with $\partial\Sigma = T$

then we claim there is a Legendrian L with $tb(L)$ negative such that the transverse push-off of L is T i.e. $L_+ = T$

from Section III we know

$$sl(T) = tb(L) - r(L)$$

$$\therefore sl(T) \leq -\chi(\Sigma) \text{ from above}$$

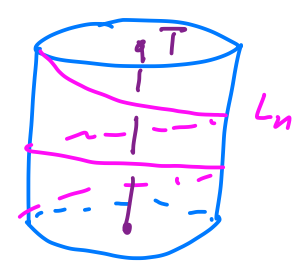
now for the claim

by Th$^{\underline{m}}$ II.3, T has a neighborhood N contactomorphic to $N_\varepsilon = \{(r,\theta,\phi) \mid r \leq \varepsilon\}$ in $(\mathbb{R}^2 \times S^1, \xi = \ker(d\phi + r^2 d\theta))$.

note $T_c = \{(r,\theta,\phi) \mid r = c\}$ has linear characteristic foliation of slope $-\frac{1}{c^2}$

choose $n \in \mathbb{N}$ s.t. $\frac{1}{\sqrt{n}} < \varepsilon$ then $T_c \subset N_\varepsilon \cong N$ has foliation of slope $-n$

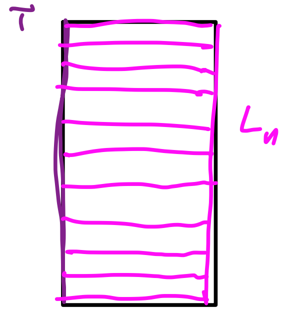
let L_n be a leaf of $(T_c)_\xi$



note L_n is smoothly isotopic to T

exercise: $tb(L_n) = -n$

L_n and T cobound an annulus ξ with T_ξ



so $(L_n)_t = T$

this finishes the proof ▦

exercise: With notation above $L_{n+1} = S_-(L_n)$

B. Simple classification results

all our classification results will rely on

Thm (Eliashberg): any two tight contact structures on B^3 that induce the same characteristic foliation on ∂B^3 are isotopic through contact structures

we will not prove this theorem but use it in our other classification results

Eliashberg proved this by studying families of foliations on S^2

Giroux gave a proof using convex surfaces

Thm 3: up to isotopy, there is a unique tight contact structure on S^3

Proof:

we know from Bennequin (or Eliashberg-Gromov) that S^3 admits a tight contact structure

now suppose ξ, ξ' are two thight contact structures on S^3

fix a point $p \in S^3$ there is an isotopy ϕ_t of S^3 such that

$\phi_t(p) = p$ for all t and

$d(\phi_1)_p(\xi') = \xi$

so $d(\phi_t)(\xi')$ is an isotopy of ξ' so that $\xi' = \xi$ at p

the proof of Darboux's theorem, Thm II.2, says we can now find an isotopy of ξ' so that $\xi' = \xi$ on a nbhd U of p

now let S^2 be a sphere in U that bounds p

note $S^3 \setminus S^2 = B^3_{in} \cup B^3_{out}$ where $B^3_{in} \subset U$

we have $\xi = \xi'$ on U so $S^2_{\xi} = S^2_{\xi'}$

Eliashberg's result above says ξ and ξ' are isotopic on B_{out}

after this isotopy $\xi = \xi'$ on all of S^3

Thm 4:

$\boxed{\text{upto isotopy there is a unique tight contact structure on } S^1 \times S^2}$

Proof:

(0-h) $\cup$ (1-h) is a Stein 4-manifold by Eliashberg-Gompf thm in Section V

(0-h) $\cup$ (1-h) $= S^1 \times D^3$

so $\partial(S^1 \times D^3)$ is $S^1 \times S^2$ with a tight contact structure.

now suppose ξ_0, ξ_1 are two tight contact structures on $S^1 \times S^2$

let $S_i = \{pt\} \times S^2$ in $S^1 \times S^2$

we can isotop S_i so it is convex in ξ_i. (Th^m VII.5)

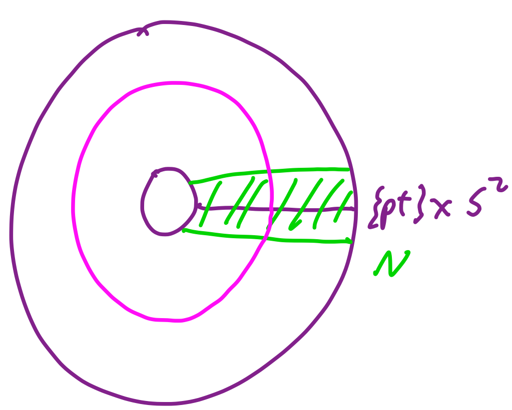
by Giroux criterion (Th^m VII.10) we know Γ_{S_i} is a simple closed curve

by Giroux flexibility (Th^m VII.6) we can assume isotop S_1 so that $(S_1)_{\xi_1}$ is the same as $(S_0)_{\xi_0}$ (i.e. there is some identification of S_1 with S_0 so this is true)

now we can isotop S_1 to S_0 so that $(S_1)_{\xi_1}$ is taken to $(S_0)_{\xi_0}$ (and extend to an ambient isotopy)

push ξ_1 forward by this isotopy and we see that, after isotoping ξ_1, we can assume $S_0 = S_1$ and $(S_0)_{\xi_0} = (S_1)_{\xi_0}$

by Th^m II.5 we may further isotop ξ_1 so that ξ_0 and ξ_1 agree on a neighborhood $N = I \times S^2$ of S_0



now let L_i be a Legendrian knot isotopic to $S^1 \times \{pt\}$

we can assume $L_0 = L_1$ in N

after stabilizing L_0 or L_1 we can assume their contact framings are the same

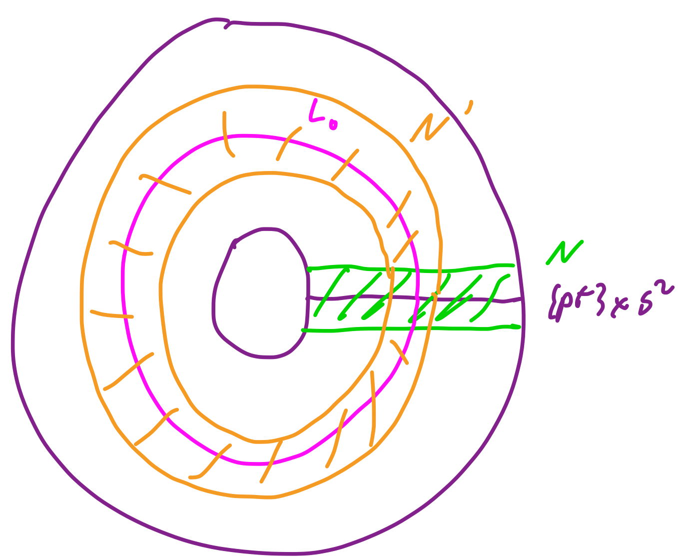
thus there is an isotopy L_1 to L_0 and extend this to an ambient isotopy of $S^1 \times S^2$

pushing ξ_1 forward by this isotopy we can assume $L_0 = L_1$

a further isotopy of ξ_1 will arrange $\xi_0 = \xi_1$ along L_0

now the proof of Th^{m} II.4 says we can isotop ξ_1 so that $\xi_0 = \xi_1$ in a neighborhood N' of L_0

(moreover this isotopy is fixed on N)



so $\xi_0 = \xi_1$ on $N \cup N'$

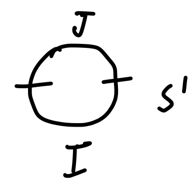
note: $S^1 \times S^2 - (N \cup N')$

$= \left((S^1 \times S^2) - (I \times S^2) \right) - S^1 \times D^2$

$= (S^1 \times (S^2 - D^2)) - (I \times (S^2 - D^2))$

$= (S^1 \times D^2) - (I \times D^2)$ different D^2

$= J \times D^2$ where



$= B^3$

let S^2 be a sphere in $N \cup N'$ that bounds a ball B' in $S^1 \times S^2$

we know $\xi_0 = \xi_1$ near S^2 $\therefore$ $S^2_{\xi_0} = S^2_{\xi_1}$

now Eliashberg's classification on B^3 says we can further isotop ξ_1 to ξ_0 on B^3

$\therefore$ $\xi_0 = \xi_1$ on all of $S^1 \times S^2$!

Thm 5: let $\mathcal{F}$ be a singular foliation on $\partial(S^1 \times D^2)$ that is divided by two parallel curves of slope n (that is each curve is homologous to $[S^1 \times \{p\}] + n[\{q\} \times \partial D^2]$)

Then upto isotopy there is a unique tight contact structure on $S^1 \times D^2$ inducing $\mathcal{F}$ on $\partial(S^1 \times D^2)$

Proof: we first show that there is some such tight contact structure.

to this end consider the $\mathbb{R}^3/_{(x,y,z)\sim(x+1,y,z)}$ with $\xi = \ker(dz - y\,dx)$

its universal cover is $\mathbb{R}^3$ with standard contact structure

so is tight by Bennequin

$S = \{(x,y,z) : x^2+y^2 \leq \varepsilon^2\}$ is a solid torus

$v = z\frac{\partial}{\partial z} + y\frac{\partial}{\partial y}$ is a contact vector field $\pitchfork \partial S$

it induces the dividing curves $\Gamma = \{z = \pm\varepsilon\}$ on ∂S

by Giroux realization we can assume $\mathcal{F}$ is $(\partial S)_\xi$ if $n=0$

for $n \neq 0$ $\exists$ a diffeomorphism of S sending Γ to

curves of slope n

so $\exists$ at least one tight structure as in the theorem.

now assume ξ, ξ' are two tight contact structures on $S^1\times D^2$

inducing $\mathcal{F}$ on $\partial(S^1\times D^2)$

Th$^{\text{m}}$ II.5 says we can isotop ξ' to agree with ξ in

a nbhd N of $\partial(S^1\times D^2)$

we can find a torus T in N parallel to $\partial(S^1\times D^2)$

that is convex with the same dividing set as $\partial(S^1\times D^2)$

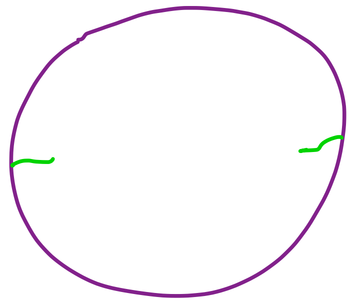
now isotop T so that T is in standard form with

ruling slope ∞ (in both $\xi = \xi'$ on N)

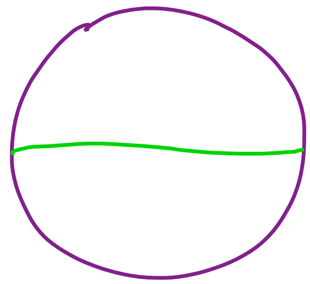
let $D = D'$ be a disk in $S^1\times D^2$ with $\partial D = \partial D' =$ ruling curve

on T (and interior in solid torus S that T bounds)

note the disks intersect Γ_T twice so lemma VII.11 says the dividing set on $D=D'$ in ξ and ξ' near ∂D is



now the Giroux criterion says Γ_D has no closed curves thus Γ_D (and $\Gamma_{D'}$) is isotopic to



using Giroux flexibility we can isotop D' so that $D'_{\xi'}$ is the same as D_ξ (under natural identification of D with D') and isotopy fixed near ∂D (since $\xi=\xi'$ near ∂D)

thus there is a smooth isotopy of D' to D taking $D'_{\xi'}$ to $D_{\xi'}$

extend to an ambient isotopy and push ξ' forward by it

thus we can assume $\xi=\xi'$ on N and $D=D'$ and ξ, ξ' induce the same foliation on D

again the proof of Thm II.5 says we can further isotop ξ' to agree with ξ in a nbhd N' of D

note $(S^1\times D^2)-(N\cup N')$ is a 3-ball

let S^2 be a sphere in $N\cup N'$ bounding a 3-ball B s.t. $N\cup N'\cup B$ is $S^1\times D^2$

since $S^2_\xi = S^2_{\xi'}$, Eliashberg's classification of tight structures on B^3 says we can isotop ξ' to agree with ξ on B^3

thus ξ' can be isotoped to agree with ξ on all of $S^1\times D^2$! $\boxplus$

exercise:

Show that if ξ is a tight contact structure on $S^1\times D^2$ with convex boundary having 2 dividing curves of slope $n\in\mathbb{Z}$ then ~~in~~ for any rational $s\leq n$ there is a torus $T\subset S^1\times D^2$ that is isotopic to $\partial(S^1\times D^2)$ and convex with 2 dividing curves of slope s

moreover, if $s<n$ then we could find such a T with linear foliation of slope s

Hint: use Th$^{\underline{m}}$ above and consider the model

$$(S^1\times\mathbb{R}^2, \ker(d\phi+r^2d\theta))$$

and recall how to perturb a linearly foliated torus to a convex torus

C. Isotopy classes of contact structures

we denote the set of isotopy classes of tight contact structures on a closed manifold M by

$$\mathrm{Tight}(M)$$

so theorems above can be restated as

$$|\mathrm{Tight}(S^3)| = |\mathrm{Tight}(S^1\times S^2)| = 1$$

now if M has boundary the if we fix a singular foliation $\mathcal{F}$ on ∂M then we denote the set of isotopy classes of tight contact structures on M inducing $\mathcal{F}$ on ∂M by

$$\mathrm{Tight}(M;\mathcal{F})$$

if Γ is a collection of curves on ∂M and $\mathcal{F}$ is any singular foliation on ∂M divided by Γ the we denote the set of isotopy classes of tight contact structures on M inducing $\mathcal{F}$ on ∂M by

$$\mathrm{Tight}(M;\Gamma)$$

lemma 6:

If $\mathcal{F}_1, \mathcal{F}_2$ are two singular foliation that are both divided by Γ then there is a one-to-one correspondence between $\mathrm{Tight}(M;\mathcal{F}_1)$ and $\mathrm{Tight}(M;\mathcal{F}_2)$

exercise: Prove this using Giroux realization thus $\text{Tight}(M;\Gamma)$ is well-defined without specifying $\mathcal{F}$

the theorems above say

$$|\text{Tight}(B^3;\Gamma)|=1$$

for Γ a connected curve

and

$$|\text{Tight}(S^1\times D^2;\Gamma)|=1$$

for Γ a two curves of slope n

finally if ∂M is a torus and s is a slope on ∂M

then

$$\text{Tight}(M;s)$$

denotes $\text{Tight}(M;\Gamma)$ where Γ consists of two curves of slope s

$$\text{so } |\text{Tight}(S^1\times D^2;n)|=1$$

if $\partial M=T_1\cup\ldots\cup T_n$, let $s_1,\ldots s_n$ be slopes on a torus

$\text{Tight}(M;s_1,\ldots,s_n)$ denotes $\text{Tight}(M;\Gamma)$

where Γ on T_i consists of two curves of slope s_i